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COMMENT

Supersymmetry and the three-dimensional isotropic oscillator problem

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Abstract. By transforming the radial equation of the isotropic oscillator to the Coulomb form, we show that the energy degeneracy of the three-dimensional oscillator may be understood on account of supersymmetry.

Recently, the one-dimensional supersymmetric construction procedure of Witten [1] has been applied to the radial problem. For the Coulomb potential, it has been observed [2] that the doubly degenerate supersymmetric energy spectrum may be associated with the well known (ns–np) degeneracy of the hydrogen atom.

The authors of [2], however, have not considered the related three-dimensional isotropic oscillator problem whose radial equation could also be subjected to a supersymmetric treatment. In this comment, we show that the energy degeneracy of the isotropic oscillator may also be explained on account of supersymmetry.

Consider the Schrödinger equation of the isotropic oscillator:

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{1}{4}r^2\right)\Phi_{nl}=E_n\Phi_{nl} \tag{1}$$

where l is the angular momentum quantum number and Φ_{nl} are the eigenfunctions corresponding to the energy eigenvalues E_n. It may be noted that the energy levels of the isotropic oscillator are separated by two units of l:

$$E_n=n+\tfrac{3}{2} \qquad n=l,\, l+2,\, l+4,\, \ldots. \tag{2}$$

A change of variable from r to $\rho=r^2$ transforms [3] (1) into the following form:

$$\left(-\frac{d^2}{d\rho^2}-\frac{3}{16\rho^2}+\frac{l(l+1)}{4\rho^2}+\frac{1}{16}-\frac{E_n}{4\rho}\right)\Psi_{nl}(\rho)=0 \tag{3}$$

where $\Psi_n(\rho)=\sqrt{r}\,\Phi_{nl}(r)$. One can do some rearranging to express (3) as

$$\left(-\frac{d^2}{d\rho^2}+\frac{\lambda(\lambda+1)}{\rho^2}+\frac{1}{16}-\frac{E_n}{4\rho}\right)\Psi_{nl}(\rho)=0 \tag{4}$$

where the parameter λ is related to l by

$$\lambda=\tfrac{1}{4}(2l-1). \tag{5}$$

Clearly, for *fixed n*, (4) corresponds to the Schrödinger equation for a Coulomb potential with angular quantum number λ.

The crucial point to realise is that a change of λ by one unit in (5) generates a change in l by two units. From [2] we know that the supersymmetry for the Coulomb case implies energy degeneracy between partner states with angular quantum numbers differing by one unit. Here we find that given this supersymmetric interpretation of the ns–np degeneracy of the Coulomb problem (i.e. between states labelled by angular quantum numbers λ and $\lambda+1$), the l and $l+2$ degeneracy of the isotropic oscillator is just a straightforward consequence. It may be noted again that in interpreting (5), we have kept n fixed. Actually, one can get rid of any n dependence in (2) by expressing E_n directly in terms of l:

$$E_l = l + \tfrac{3}{2} \qquad l = 0, 2, 4, \ldots. \tag{6}$$

We conclude with the following remarks.

(i) The transformation from the Coulomb problem to the isotropic oscillator is quite well known [3]. In [3] other forms of the potentials may be found which may be transformed to the Coulomb potentials.

(ii) Instead of transforming the Schrödinger equation from the isotropic oscillator to the Coulomb potential, if we had naively applied Witten's procedure to the isotropic oscillator problem, we would have obtained energy degeneracy between states with angular numbers l and $l+1$, respectively. However, this would not have been consistent with (6) which predicts degeneracies corresponding to a separation of two units of l. Perhaps this is why the isotropic oscillator problem was not considered in [2]. Nonetheless, one can still explain equations (2) or (6) by going over to a supersymmetric construction on the full line. We have discussed this and related issues in a recent article [4].

Note added in proof. After the submission of this comment for publication, we discovered a related work by Kostelecky *et al* [5] in which the relationship between the radial equations of the d-dimensional hydrogen atom and the D-dimensional harmonic oscillator has been explicitly constructed.

References

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